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A disorder solution for a cubic Ising model

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Abstract. The exact expression of the partition function of a three-dimensional cubic Ising model, with nearest-neighbour interactions, is given, when a certain relation between the coupling constants of the model is satisfied. This disorder solution is then compared with a partially resummed high-temperature expansion of the partition function of the model. The constraints on this expansion, which result from the existence of the disorder solution and from the inversion relation are discussed.

1. Introduction

Presently, disorder solutions (Stephenson 1970) are most often the only exact and quantitative piece of information one can obtain on the partition functions of three-dimensional models (Welberry and Miller 1978). Among these models, the Ising model on a cubic lattice deserves particular attention, because of its simplicity. Making use of a simple criterion, which will be fully developed in another paper (Jaekel and Maillard 1985), we exhibit here a disorder solution, having codimension one in the parameter space of a cubic Ising model with three parameters (interactions between nearest neighbours only). The model will be more precisely described in the following.

The very simple character of this solution (which is rational on that disorder variety) leads us to study how this exact expression agrees with already known information, available under the form of diagrammatic expansions. For that purpose, the second part of the paper is devoted to the computation of a partially resummed high-temperature expansion of the partition function. Finally, and with the help of the other exact piece of information on this model, the inversion relation, we analyse the constraints on the expansion (Baxter 1980), that result from the existence of this solution, keeping in view a complete determination of the partition function.

2. Geometrical derivation

We shall consider a three-dimensional cubic Ising model, with interactions between nearest neighbours. There will be three different coupling constants, arranged in a

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staggered way, as indicated by figure 1, which represents the elementary cell generating the whole lattice. One will note in particular that in two directions, the plane sections correspond to checkerboard lattices (Domb and Green 1972), without frustration, while in the remaining direction, the plane sections correspond to particular staggered lattices, with full frustration (see figure 2). The partition function per site Z of this model will be defined by:

$$Z^N(K, K', L) = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \exp(K_{ij}\sigma_i\sigma_j) \qquad \sigma_i \in \mathbb{Z}_2$$

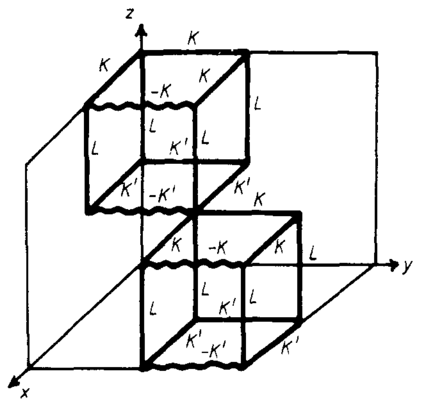


Figure 1. Elementary cell generating the lattice.

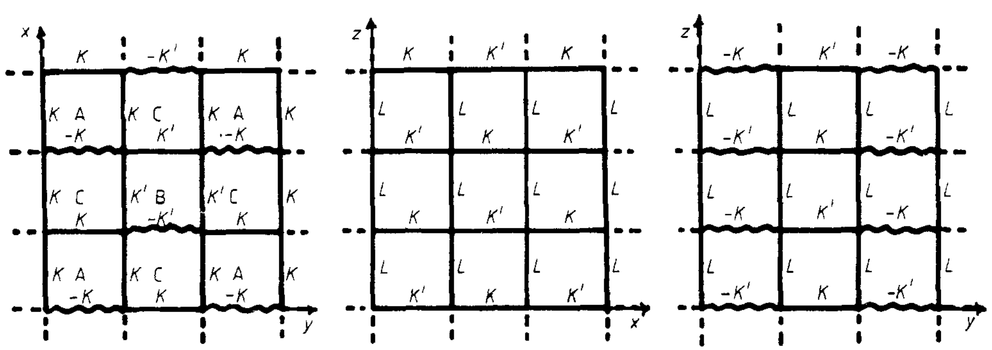


Figure 2. Plane sections of the lattice in the different directions.

where K_{ij} alternatively stands for $\pm K$, $\pm K'$, or L, according to figures 1 and 2, and i, j denote the different sites (of total number N). We shall exhibit an exact solution for the partition function per site, for a certain relation between the three parameters, making use of a criterion which is made explicit and developed elsewhere (Jaekel and Maillard 1985). According to this criterion, one obtains the relation between the parameters, by requiring the following property: when summed over the four spins of the upper face, the Boltzmann weight for an elementary cube does not depend any longer on the four spins of the lower face (see figure 3):

$$W(K, K', L, s) = \sum_{(\sigma)} \exp[2KX(\sigma) + L\sigma \quad s + 2K'X(s)]$$

$$X(\sigma)=\tfrac{1}{2}(-\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1) \qquad (X(\sigma)^2=1)$$

$$\sigma\cdot s=\sum_{i=1}^{4}\sigma_i s_i.$$

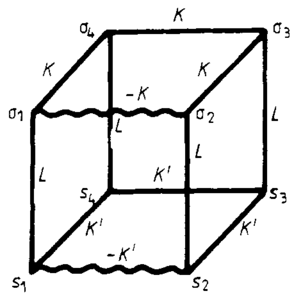


Figure 3. Spins on an elementary cube.

Introducing high-temperature variables:

$$T=\tanh 2K \qquad T'=\tanh 2K' \qquad \tau=\tanh L$$

this condition can be rewritten:

$$W(K,K',L,s)=\frac{1}{(1-T^2)^{1/2}}\,\frac{1}{(1-T'^2)^{1/2}}\,\frac{1}{(1-\tau^2)^2}\,\omega(T,T',\tau,s)$$

$$\omega(T,T',\tau,s)=\sum_{(\sigma)}[1+TX(\sigma)]\left[\prod_{i=1}^{4}(1+\tau\sigma_i s_i)\right][1+T'X(s)]$$
$$=2^4[1+\tau^2TT'+(\tau^2T+T')X(s)]$$

must be independent of (s), that is:

$$\tau^2T+T'=0 \tag{1}$$

so that the Boltzmann weight of the cube becomes:

$$W_{|\tau^2T+T'=0}=W^*=\frac{2^4}{(1-\tau^2)^2}\,\frac{(1-\tau^4T^2)^{1/2}}{(1-T^2)^{1/2}}=\frac{2^4}{(1-\tau^2)^2}\,\frac{(1-T'^2)^{1/2}}{[1-(T'^2/\tau^4)]^{1/2}}. \tag{2}$$

Let us now consider the whole lattice, with the following boundary conditions: on the upper (xy) layer, only the interactions K which correspond to the squares of type A of figure 2 are present. It then appears that the summation over all the spins of the upper layer, when condition (1) is realised, leads one back to the previous situation, but this time on the next layer, and for the squares of type B. This allows one to iterate this procedure, until all the spins of the whole cubic lattice have been summed over. The partition function per site of this lattice is then given by the simple formula:

$$Z(K,K',L)_{|\tau^2T+T'=0}=W^{*1/4}. \tag{3}$$

Clearly, the particular boundary conditions introduced here, to be compared with the standard periodic boundary conditions, do not modify the partition function per site in the thermodynamic limit, at least in the high-temperature part of the physical domain (disorder domain). The occurrence, in the heart of the phase diagram of a purely three-dimensional model, of a large algebraic variety (co-dimension one), where the partition function simplifies to give a zero-dimensional partition function, constitutes a remarkable phenomenon. It then appears interesting to try to integrate this valuable piece of information within the present knowledge about the partition function, i.e. expansions, and at best partially resummed expansions, and to look for the consequences of this exact solution on the analytical behaviour of the partition function.

3. Diagrammatic expansion

3.1. Partially resummed expansion

In this section we shall determine the normalised partition function per site Λ in a particular diagrammatic expansion of the type which has already been used for the standard cubic anisotropic Ising model (Jaekel and Maillard 1982). It is related to the partition function per site by the following formula:

$$\Lambda(t, t', \tau) = \frac{Z(K, K', L)}{2 \cosh K \cosh K' \cosh L} \qquad (t = \tanh K, t' = \tanh K', \tau = \tanh L). \tag{4}$$

To determine the latter, we shall resum a high-temperature expansion in one of the three directions (here resummation in τ), and compute it up to order four in t and t'. The different terms in the expansion are given by all the closed diagrams one can draw on the lattice. The resummation in τ and the particular staggered structure of the lattice lead us to consider three different types of diagrams:

(i) The two-dimensional diagrams which lie in the (xz) and (yz) planes of figure 2; these identify with the diagrams of the two-dimensional checkerboard Ising model (Domb and Green 1972), and will be called type (I) diagrams (contribution $\ln \Lambda_{\rm I}$). As the negative bonds must always occur in an even number, their signs do not modify any contribution so that the total sum of type (I) diagrams can be directly written (Jaekel and Maillard 1984):

$$\begin{aligned}\ln \Lambda_{\rm I} = {} & [\tau^2/(1-\tau^4)][\tau^2(t^2+t'^2)+2tt'] + 2\frac{\tau^2(1+\tau^2)}{(1-\tau^4)^3}[\tau^2(t^2+t'^2)+(1+\tau^4)tt']^2 \\ & + [\tau^4/(1-\tau^4)^3][\tau^4(-\tfrac{7}{2}+\tfrac{1}{2}\tau^4)(t^4+t'^4) - 4\tau^2(2+\tau^4)(t^3t'+t'^3t) \\ & - (5+\tau^4)(1+2\tau^4)t^2t'^2].\end{aligned} \tag{5}$$

(ii) The strictly three-dimensional diagrams (of order four in t and t') which lie in the particular vertical tubes standing over the type (A) or (B) squares of the (xy) plane (see figure 2); these will be called type (II) diagrams and their total contribution will be denoted by $\ln \Lambda_{\rm II}$.

(iii) The strictly three-dimensional diagrams (of order four in t and t') which lie in the particular vertical tubes standing over the type (C) squares of the (xy) plane, and which we shall call type (III) diagrams. Their contribution will be denoted by $\ln \Lambda_{\rm III}$.

These last two classes of diagrams have already been displayed in the study of the standard anisotropic three-dimensional cubic Ising model, but their respective contributions will differ here, in order to account for the specific couplings in this new model. In particular, it appears as a consequence of the negative bonds, that a good number of diagrams cancel each other. This occurs separately for type (II) and (III) diagrams (see figure 4). Indeed, it is easy to see that a diagram with the opposite contribution can be obtained by reflecting the lower basis with respect to its diagonal, so that the combinatorial factors for diagrams of opposite contributions are exactly equal. However, the remaining diagrams must be considered separately according to their type: *Type (II) diagrams.* The diagrams and their respective contributions are given by figure 5. Let us note that the rational functions in τ, which are not easy to determine directly, are better obtained by combining the knowledge of the diagrammatic expansion up to order six in τ with the following two simple limits: at $t = t'$, the contribution of each diagram is equal to the corresponding one of the standard model (with an overall change of sign, except for the disconnected diagrams); moreover at $t = 0$, the same identification occurs but this time with τ^2 replacing τ^4. The total sum

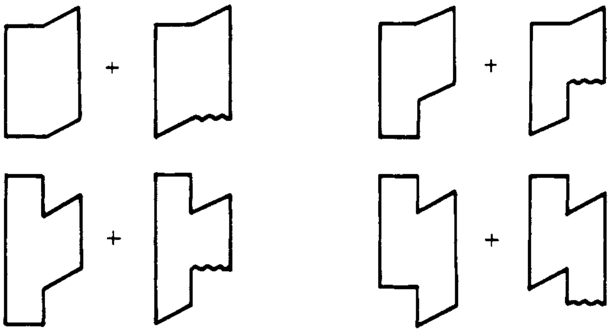
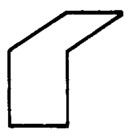
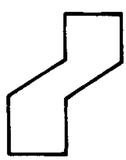


Figure 4. Diagrams cancelling each other.

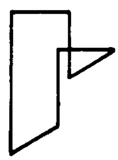
$\frac{1}{4}(t^4 + t'^4)$



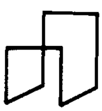
$$\frac{2\tau^2(t^3t' + t'^3t) + 2\tau^4(t^4 + t'^4)}{1 - \tau^4}$$



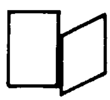
$$\frac{2\tau^4t^2t'^2 + 2\tau^6(t^3t' + t'^3t) + \tau^8(t^4 + t'^4)}{(1 - \tau^4)^2}$$



$$\frac{4\tau^4t^2t'^2 + 4\tau^6(t^3t' + t'^3t) + 2\tau^8(t^4 + t'^4)}{(1 - \tau^4)^2}$$



$$[\tau^4(1 + 5\tau^4)t^2t'^2 + 2\tau^6(1 + \tau^4)(t^3t' + t'^3t) + \tfrac{1}{2}\tau^8(1 + \tau^4)(t^4 + t'^4)][(1 - \tau^4)^{-3}]$$



$$[2\tau^4(1 + 5\tau^4)t^2t'^2 + 4\tau^6(1 + \tau^4)(t^3t' + t'^3t) + \tau^8(1 + \tau^4)(t^4 + t'^4)][(1 - \tau^4)^{-3}]$$

Figure 5. Type (II) diagrams and contributions.

is then given by:

$$\ln \Lambda_{\mathrm{II}} = -\frac{1+5\tau^4+5\tau^8+\tau^{12}}{(1-\tau^4)^3}\,\frac{(t^4+t'^4)}{4} - 2\tau^2\,\frac{(1+4\tau^4+\tau^8)}{(1-\tau^4)^3}\,(t^3t'+t'^3t) - 9\,\frac{\tau^4(1+\tau^4)}{(1-\tau^4)^3}\,t^2t'^2. \tag{6}$$

Let us also note that this contribution can be obtained in another indirect way; indeed, at this order (four in t and t') the partition function of the model studied here has the same three-dimensional part as that of a lattice reduced to a single tube (quasi one-dimensional lattice). The exact computation of the latter is easily accomplished and is given in the appendix, showing that the extracted three-dimensional part effectively coincides with expression (6).

Type (III) diagrams. The diagrams and their contributions are given by figure 6. Their total contribution is

$$\ln \Lambda_{\mathrm{III}} = -\frac{3}{2}\,\frac{\tau^4(1+\tau^4)}{(1-\tau^4)^3}\,(t^4+t'^4) - 2\tau^2\,\frac{(1+4\tau^4+\tau^8)}{(1-\tau^4)^3}\,(t^3t'+t'^3t) - \frac{1}{2}\,\frac{1+17\tau^4+17\tau^8+\tau^{12}}{(1-\tau^4)^3}\,t^2t'^2. \tag{7}$$

For these diagrams, only the $t=t'$ limit can be used, showing that the respective contributions identify with those of the standard anisotropic cubic Ising model (with a change of sign, except again for the disconnected diagrams). Summing over the contributions of the three classes, one gets the required expansion:

$$\ln \Lambda = \ln \Lambda_{\mathrm{I}} + \ln \Lambda_{\mathrm{II}} + \ln \Lambda_{\mathrm{III}}. \tag{8}$$

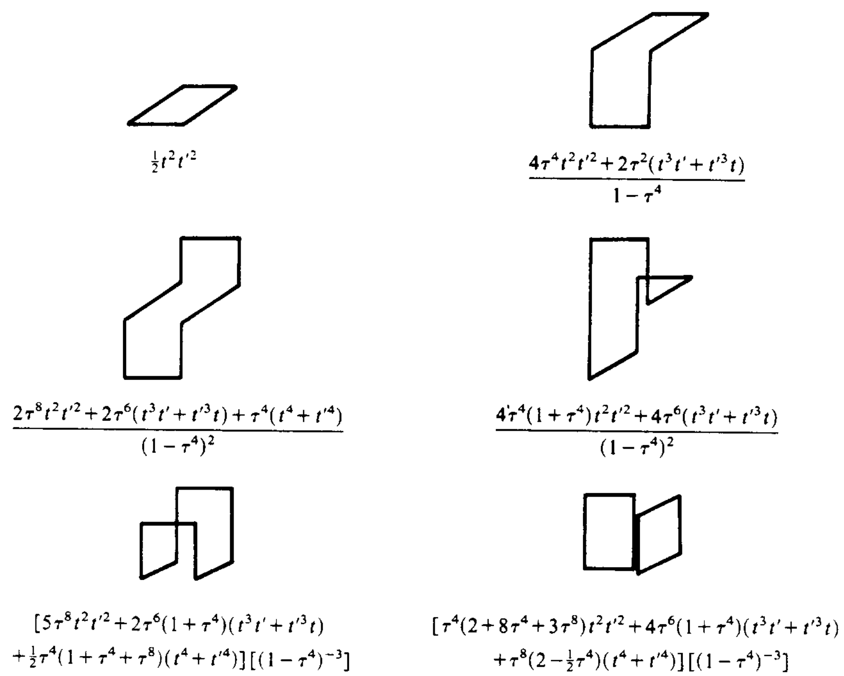


Figure 6. Type (III) diagrams and contributions.

3.2. *Discussion*

3.2.1. Inversion relation and disorder solution. A straighforward modification of the reasoning which allowed one to obtain the inversion relation on the standard anisotropic cubic Ising model (Jaekel and Maillard 1982) shows that the partition function of the model studied here must also satisfy the following inversion relation:

$$Z(K, K', L)Z(-K, -K', L+i\pi/2) = 2i \sinh 2L$$

or

$$\ln \Lambda(t, t', \tau) + \ln \Lambda(-t, -t', \tau^{-1}) = \ln(1-t^2) + \ln(1-t'^2). \tag{9}$$

Indeed, this relation is easily seen to be satisfied by the expansion of $\ln \Lambda$, up to order four (5, 6, 7, 8). Moreover, the inversion relation also allows one to put constraints on the different terms which build $\ln \Lambda$ (8). First, it is well known that type (I) diagrams, which correspond to the expansion of the partition function of the checkerboard two-dimensional Ising model satisfy an inversion relation:

$$\ln \Lambda_{\rm I}(t, t', \tau) + \ln \Lambda_{\rm I}(-t, -t', \tau^{-1}) = \ln(1-t^2) + \ln(1-t'^2).$$

Then, type (II) diagrams can be extracted from the partition function of the single tube model considered in the appendix, which verifies the same inversion relation. They satisfy

$$\ln \Lambda_{\rm II}(t, t', \tau) + \ln \Lambda_{\rm II}(-t, -t', \tau^{-1}) = 0.$$

Finally, from the global inversion relation (9), type (III) diagrams must also satisfy

$$\ln \Lambda_{\rm III}(t, t', \tau) + \ln \Lambda_{\rm III}(-t, -t', \tau^{-1}) = 0.$$

The validity of the disorder solution (2, 3) is also easily verified, up to order four, on expansion (5, 6, 7, 8). However, as in the case of the inversion relation, it can also be used to put separate constraints on the different parts which build $\ln \Lambda$. From the already known disorder solution for the checkerboard two-dimensional Ising model (Baxter 1984) it results that type (I) contributions verify

$$\ln \Lambda_{\rm I}(-t'/\tau^2, t', \tau) = \ln(1-t'^2) = -t'^2 - \tfrac{1}{2}t'^4 + o(t'^4).$$

Noting that, at this order, the disorder relation (1) is

$$t = -t'/\tau^2 + (1-\tau^{-4})t'^3/\tau^2 + o(t'^4) \qquad (T = 2t/(1+t^2),\ T' = 2t'/(1+t'^2)),$$

and that

$$\ln \Lambda_{\rm I|\tau^2 T + T' = 0} = \ln \Lambda_{\rm I}(-t'/\tau^2, t', \tau) + o(t'^4),$$

one gets

$$\ln \Lambda_{\rm I|\tau^2 T + T' = 0} = -t'^2 - \tfrac{1}{2}t'^4 + o(t'^4).$$

As is shown in the appendix, the verification of the disorder solution by the partition function of the single tube model also implies (up to order four), on type (II) contributions:

$$\ln \Lambda_{\rm II|\tau^2 T + T' = 0} = (1-\tau^{-8})t'^4/4 + o(t'^4).$$

Finally, from the global disorder solution (3, 4), type (III) diagrams must satisfy:

$$\ln \Lambda_{\rm III|\tau^2 T + T' = 0} = o(t'^4).$$

Let us just remark that, for the inversion relation and for the disorder solution also, the $1-\tau^4$ singularities which occurred in the diagram resummation exactly cancel, in order to give the simple rational expressions entering both equations.

3.2.2. Constraints. From the previous verifications it appears that these two remarkable and exact properties (the inversion relation and the disorder solution) constrain rather strongly the expansion of the partition function. One can then hope to determine this expansion, by combining these two exact properties with a qualitative knowledge of the diagrammatic expansion. Indeed, the presence of a unique singularity $(1-\tau^4)^3$ at this order, allows one to write:

$$\ln \Lambda_{II} + \ln \Lambda_{III} = [P(\tau^2)/(1-\tau^4)^3](t^4+t'^4) + [Q(\tau^2)/(1-\tau^4)^3](t^3t'+t'^3t) + [R(\tau^2)/(1-\tau^4)^3]t^2t'^2 + \ldots$$

P, Q, R are polynomials which must then verify (inversion symmetry):

$$P(\tau^2) = \tau^{12}P(1/\tau^2) \qquad \text{(resp. for } Q \text{ and } R\text{)}.$$

The disorder solution then implies the following factorisation condition:

$$(1+\tau^8)P(\tau^2) - \tau^2(1+\tau^4)Q(\tau^2) + \tau^4R(\tau^2) = -(1-\tau^4)^3(1-\tau^8).$$

Collecting the different pieces of information shows that the expansion is not yet fully determined. One could then envisage completing the information with a closer examination of the different terms which enter the polynomials P, Q, R, and which come from different diagrams (for instance, no τ^4 and τ^8 terms are present in Q). Moreover, the nature of the singularities which must occur in the expansion is not even quite clear (are there only $(1-\tau^4)$ singularities?). A clarification of this problem appears as a necessary step towards a systematic determination of the partition function, using this method.

4. Conclusion

The partition functions of three-dimensional models, like the Ising model studied here, are still poorly understood objects. Hence the knowledge of the latter on some remarkable manifold, like the disorder variety, is a valuable piece of information.

The simultaneous use of the disorder solution and of the inversion relation has made it possible to derive a maximum number of constraints to be imposed on an adapted diagrammatic expansion. Still more constraints would be obtained if the inversion were associated with spatial symmetries of the model, in order to generate a larger group of symmetries. However, to get symmetries which do not commute with the inversion, one needs to increase the number of parameters in the model (i.e. different couplings on the twelve bonds of the elementary cube). Simple disorder solutions of the type studied here can still be found, though they correspond to varieties of higher co-dimensions (Jaekel and Maillard 1985). On the other hand, in the most interesting cases, the group can then be infinite (for Potts models, like the two-dimensional checkerboard one, (Jaekel and Maillard 1984)) and generate, from the disorder variety, an infinite number of transformed varieties.

The regular analytic character of the expression taken by the partition function on a remarkable submanifold, (here a rational expression), also leads in general to constraints on its analytical behaviour in its whole physical domain. In particular, either the critical manifold(s) will avoid the disorder variety, or, at their intersection, there will result severe constraints on the singular part of the partition function. This can then be arranged in several ways. In the case of a single critical manifold, the amplitude of the corresponding singular part will vanish at the intersection (for a constant exponent) or (for a varying one) the exponent will there become an integer. In the case of several critical manifolds and a disorder variety intersecting simultaneously (for instance along a tricritical subvariety), this will lead to the cancellation of some combination of the amplitudes of the respective singular parts, or even to the cancellation of all the amplitudes (different exponents), or else to the trivialisation of the varying exponents. It can be noted that the presently known cases of such regular exact solutions correspond either to no intersection inside the physical domain (two-dimensional triangular Potts (Rujan 1984) or Ising with a field (Verhagen 1976), or to a tricritical intersection (Sherrington-Kirkpatrick model, for which Nishimori (1981) has developed in the same way the consequences of such a regular solution for the phase diagram).

Appendix. Single tube partition function

We shall consider here a lattice which is similar to the one introduced in § 2, but this time restricted to a single tube (see figure 7). Its partition function per site $\bar{Z}$ is easily computed. With the same notations as in § 2, it can be written:

$$\bar{Z}(K, K', L) = 2(\cosh K \cosh K')^{1/2}(\cosh L)\bar{\Lambda} \qquad (10)$$

$$\bar{\Lambda} = [(1-t^4)(1-t'^4)]^{1/8}\bar{\omega}$$

$$\bar{\omega}^{8N} = \sum_{\{\sigma^n, s^n\}} \prod_{n=1}^{N} \bar{\omega}_n^{n+1}$$

$$\bar{\omega}_n^{n+1} = [1 + TX(\sigma^n)]\left[\prod_{i=1}^{4}(1+\tau\sigma_i^n s_i^n)\right][1 + T'X(s^n)]\left[\prod_{i=1}^{4}(1+\tau s_i^n \sigma_i^{n+1})\right].$$

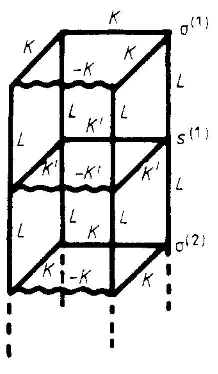


Figure 7. Single tube lattice.

Denoting:

$$\sum_{\substack{\{\sigma^n, s^n\} \\ n \leq m}} \prod_{n \leq m} \bar{\omega}_n^{n+1} = 2^{8m}[a_m + b_m X(\sigma^{m+1})],$$

one remarks that:

$$\begin{vmatrix} a_{m+1} \\ b_{m+1} \end{vmatrix} = M \begin{vmatrix} a_m \\ b_m \end{vmatrix} \qquad M = \begin{vmatrix} 1+\tau^2 TT' & T+\tau^2 T' \\ \tau^2(\tau^2 T + T') & \tau^2(\tau^2 + TT') \end{vmatrix}$$

so that the partition function per site is obtained from the largest eigenvalue λ of the matrix M:

$$\ln \tfrac{1}{2}\bar{\omega} = \tfrac{1}{8} \ln \lambda$$

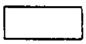
which is one of the roots of the following characteristic polynomial:

$$\lambda^2 - (1 + \tau^4 + 2\tau^2 TT')\lambda + \tau^4(1 - T^2)(1 - T'^2) = 0.$$

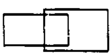
The expansion in t and t' can be directly determined, leading to the normalised partition function per site:

$$\ln \bar{\Lambda} = \frac{1}{2}\frac{\tau^2}{1-\tau^4}[\tau^2(t^2+t'^2)+2tt'] - \frac{1+5\tau^4+11\tau^8-\tau^{12}}{(1-\tau^4)^3}\frac{(t^4+t'^4)}{8}$$
$$-\frac{\tau^2(1+6\tau^4+\tau^8)}{(1-\tau^4)^3}(t^3t'+t'^3t) - 6\frac{\tau^4(1+\tau^4)}{(1-\tau^4)^3}t^2t'^2. \tag{11}$$

On the other hand this expansion is also given by the sum of all the closed diagrams lying inside the tube. These are of two kinds: strictly three-dimensional ones, which identify with the type (II) diagrams of § 3, and the following two-dimensional diagrams:

 $\dfrac{1}{2}\dfrac{\tau^2}{1-\tau^4}[\tau^2(t^2+t'^2)+2tt']$

disconnected ones:

 $-\tau^8\dfrac{(3-\tau^4)}{(1-\tau^4)^3}\dfrac{(t^4+t'^4)}{4} - \dfrac{2\tau^6}{(1-\tau^4)^3}(t^3t'+t'^3t) - \dfrac{3}{2}\dfrac{\tau^4(1+\tau^4)}{(1-\tau^4)^3}t^2t'^2.$

From these results, type (II) diagrams are easily deduced by difference.

From geometrical considerations, one can argue that the partition function per site of this model must satisfy an inversion relation. Taking the corresponding automorphic factors into account, the latter leads to the following equation in $\ln \bar{\Lambda}$:

$$\ln \bar{\Lambda}(t, t', \tau) + \ln \bar{\Lambda}(-t, -t', 1/\tau) = \tfrac{1}{2}\ln(1-t^2) + \tfrac{1}{2}\ln(1-t'^2).$$

One should note that the two-dimensional part, which also identifies with the partition function of a strip model, for which similar geometrical arguments hold, also verifies the same equation.

A straightforward transposition of the iterative procedure used in § 2, shows that for the same relation between the parameters of the model (1), the partition function per site of a single tube must also reduce to that of a cube (up to a $(\cosh L)^{1/2}$ factor):

$$\bar{Z}_{|\tau^2 T + T' = 0} = 2^{1/2}(\cosh L)^{1/2} W^{*1/8}. \tag{12}$$

Indeed, when relation (1) is inserted into equation (11), the latter becomes

$$\ln \bar{\Lambda}_{|\tau^2 T+T'=0} = -t'^2 - (1+\tau^{-8})t'^4/4 + o(t'^4)$$

which does correspond to the expansion of (10) and (12):

$$\ln \bar{\Lambda}_{|\tau^2 T+T'=0} = \tfrac{1}{8}\ln\left((1-t^4)\frac{(1-t'^2)^3}{1+t'^2}\right)_{|\tau^2 T+T'=0}$$

(note that relation (1) corresponds to a triangulation of the matrix M).

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